Day 4: Integration with a Boundary Condition and the Definite Integral

Initial Value Problems

When we integrate a function \( f(x) \) we get a family of functions \( F(x) + C \), but sometimes we are interested in one particular function. We can identify this function by specifying a specific point it passes through.

Ex. \( f(x) = 3 + 2x - x^2 \) \( F(0) = 1 \).

\[
F(x) = \int 3 + 2x - x^2 \, dx = 3x + x^2 - \frac{x^3}{3} + C
\]

This question can also be presented as \( \frac{dy}{dx} = 3 + 2x - x^2 \) \( y(0) = 1 \)

(where the unknown is not a number, but a function)

This type of problem (involving the derivative of an unknown function) is called a differential equation.

The Definite Integral

Area Interpretation

\[
\text{NET Area} = \text{Area over } x\text{-axis} - \text{Area under } x\text{-axis}
\]

\[
= A_1 + A_3 - A_2
\]

\[
\text{TOTAL Area} = A_1 + |A_2| + A_3
\]

**Notation:** Recall that \( \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x \) over some domain \([a,b]\) is approximately equal to \( \int_{a}^{b} f(x) \, dx \).

\[
\int_{a}^{b} f(x) \, dx \quad \text{is the definite integral of } \ f \text{ from } a \text{ to } b \quad \text{where } a \text{ is the lower limit, } b \text{ is the upper limit, } f(x) \text{ is the integrand}
\]
Ex. Sketch and evaluate area between curve and $x$-axis using geometric formulae.

1) \[ \int_{1}^{4} 2 \, dx \]

\[ f(x) = 2 \]

Area = $2 \times 2 = 4 \text{ units}^2$

\[ \int_{1}^{4} 2 \, dx = \left[ 2x \right]_{1}^{4} \]

\[ = 2(4) - 2(1) \]

\[ = 8 - 2 \]

\[ = 6 \]

2) \[ \int_{-1}^{2} (x + 2) \, dx \]

\[ f(x) = x + 2 \]

\[ \int_{-1}^{2} (x + 2) \, dx = \left[ \frac{x^2}{2} + 2x \right]_{-1}^{2} \]

\[ = \left( \frac{(2)^2}{2} + 2(2) \right) - \left( \frac{(-1)^2}{2} + 2(-1) \right) \]

\[ = (6) - (\frac{1}{2} - 2) \]

\[ = \frac{7.5}{2} \]

3) \[ \int_{0}^{2} (x - 1) \, dx \]

Note: Must split into two integrals. (Total area is 0).

\[ \int_{0}^{2} (x - 1) \, dx = \left[ \frac{x^2}{2} - x \right]_{0}^{2} \]

\[ = (0) - (0) \]

\[ = 0 \]

\[ \int_{0}^{2} (x - 1) \, dx = \left| \int_{0}^{1} (x - 1) \, dx \right| + \int_{1}^{2} (x - 1) \, dx \]

The Fundamental Theorem of Calculus:

If $f$ is continuous on $[a, b]$ and $F$ is the antiderivative of $f$ on $[a, b]$, then:

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) = \left[ F(x) \right]_{a}^{b} \]

Note: When calculating a definite integral, the net area is being calculated.

Refer to old examples.
Basic Properties:

i) \( \int_{a}^{a} f(x) \, dx = 0 \)

ii) \( \int_{a}^{b} f(x) \, dx = - \int_{b}^{a} f(x) \, dx \)

iii) \( \int_{a}^{b} c f(x) \, dx = c \int_{a}^{b} f(x) \, dx \)

iv) \( \int_{a}^{b} [f(x) \pm g(x)] \, dx = \int_{a}^{b} f(x) \, dx \pm \int_{a}^{b} g(x) \, dx \)

v) \( \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx \) \quad a \leq c \leq b

Examples:

1) \( \int_{1}^{2} x^2 \, dx = 0 \)

2) \( \int_{1}^{2} e^{3x} \, dx \)

3) \( \int_{0}^{\pi/4} 3 \cos(2x) \, dx \)

\[
\text{let } u = 3x \\
\frac{du}{dx} = 3 \Rightarrow \frac{du}{3} = dx \\
\int_{1}^{2} e^{u} \frac{du}{3} = \frac{1}{3} \int_{1}^{2} e^{u} \, du \\
= \frac{1}{3} \left[ e^{u} \right]_{1}^{2} = \frac{1}{3} \left[ e^{3x} \right]_{1}^{2} \\
= \frac{1}{3} \left[ e^{6} - e^{3} \right]
\]

4) \( \int_{1}^{3} (x + e^{-2x}) \, dx \)

\[
= \left[ \frac{x^2}{2} - \frac{e^{-2x}}{2} \right]_{1}^{3} \\
= (\frac{9}{2} - \frac{e^{-6}}{2}) - (\frac{1}{2} - \frac{e^{-2}}{2}) \\
= 4 - \frac{1}{2}(e^{-6} + e^{-2})
\]

5) Find value(s) of \( m \) where \( \int_{0}^{m} (2x - 4) \, dx = 5 \).

\[
\left[ x^2 - 4x \right]_{0}^{m} = 5 \\
m^2 - 4m = 5 \\
m^2 - 4m - 5 = 0 \\
(m - 5)(m + 1) = 0 \\
m = 5 \quad \text{or} \quad m = -1
\]

Suppose to be 5.
6) Find the total area of the region enclosed by \( y = x^3 - 1 \), the \( x \)-axis, and \( 0 \leq x \leq 2 \). (Note: need to consider net signed areas). (Split into two integrals and take the negative of the negative area).

\[
\int_0^1 (x^3 - 1) \, dx + \int_1^2 (x^3 - 1) \, dx
\]

Graphing Calculator Use for Definite Integrals

Go to: MATH \( \rightarrow \) fnInt()

The arguments for fnInt() are as follows:

\[
\text{fnInt( expression for function, variable used, lower bound, upperbound )}
\]

ie: To find \( \int_1^2 e^{3x} \, dx \)

\[
\text{fnInt( } e^{3x}, x, 1, 2) \rightarrow \text{ ENTER}
\]